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# Discounted optimal growth in the two-sector RSS model: a geometric investigation*

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**Abstract.** This paper initiates a comprehensive investigation of discounted optimal growth in the two-sector RSS model as a specific instance of the general theory of resource allocation associated with Brock, Gale and McKenzie. For an interval of values of a parameter $\xi$ formalizing the marginal technical rate of transformation, under zero consumption, of machines from one period to the next, we show that the optimal policy in the discounted case remains *identical* to that in the undiscounted case *irrespective* of the discount factor. For two particular cases of $\xi$ outside the said interval, we give a complete characterization of the optimal policy function, and of a variety of subsets that extend the facet notions formulated by McKenzie. Methodologically, this essay is a further rehabilitation of the geometric apparatus introduced by the authors for the undiscounted case.

**Key words:** RSS model, irreversible investment, modified golden-rule stock, golden-rule prices, value-loss lines, cycling, von-Neumann facet, McKenzie facet, $m_i$-facet, $v_i$-facet, transition dynamics, optimal policy function

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# 1. Introduction

With increased understanding into the optimal dynamics of a model due to Robinson, Solow and Srinivasan, the so-called RSS model[1], three points of methodological significance attain salience: a dramatic difference between the discrete- and continuous-time analyses, a suggestion for a unified analysis in the emergence of the undiscounted case as a guiding marker for the discounted case, and an important, if not indispensable, role for plane geometry as a complementary engine of analysis.

As regards the first issue, it is not the case that the undiscounted continuous-time analysis obtained by Stiglitz [40], and reconfirmed in [13][2], is the correct idealized limit for that presented in the discrete-time setting in [16, 45]. Put another way, it is not apparent how qualitative properties of the discrete-time dynamics can be interpreted as valid approximations to those obtained in continuous-time: policies identified by Stiglitz as optimal in one formulation, are shown to be *bad,* leave alone non-optimal, in another; and a parameter $\xi_\sigma$, formalizing the marginal rate of transformation of machines of a particular type $\sigma$ from one period to the next, and governing circumstances under which monotone dynamics are transformed into cyclical optimal behavior, stationary and damped, simply plays no role when time cannot be divided into distinguishable periods of a given length[3]. In a nutshell, the asymptotic implementation of Stiglitz' results for the RSS model remains yet to be accomplished[4].

With respect to the second issue, preliminary computations suggest it to be the case that for certain ranges of the parameter $\xi_\sigma$, optimal programs in specific examples of the RSS model are independent of the discount factor $\rho$ when it is restricted to non-degenerate and determinate intervals, and that chaotic behavior arises only when this discount factor is below a computable threshold; see [11, 17]. It appears that the RSS model offers a particularly sharp

[1] This is the terminology proposed in [16]. This paper may be referred to for variants of the model as well as for detailed references to earlier work. For the results that are antecedent to those presented here, the reader is referred to [16, 14, 15, 11, 17], [45] and [27], among other (ongoing) work.

[2] In [40], Stiglitz obtains the optimal trajectories in the undiscounted case simply by equating the discount factor to unity in formulae obtained through the use of Pontryagin's maximum principle for the discounted case. In [13], his insight is rigorously substantiated through an analysis based on the overtaking criterion and on Brock's ideas.

[3] See Examples 1 and 2 in [16]. For a complete characterization of optimal policies in a two-sector RSS model without discounting and with a linear felicity function, see [14, 15].

[4] This paragraph has benefitted from conversations with Leo Hurwicz and Roy Radner at the Tokyo Conference. Its parallel to relevant situations of games (and economies) with a continuum of agents and/or a continuum of (dispersed) information are too close to be ignored, see [18] and work subsequent to it.

instance of the fact that the unit discount factor is not a bifurcation, and that for a range of parametric values, the qualitative properties of the optimal dynamics of the discounted and undiscounted cases remain *unchanged.* All this is in keeping with the insight that the undiscounted case is an important analytical benchmark for the discounted analysis even if one does not find compelling the philosophical grounds of Ramsey and of others[5] for the use of a zero discount rate in the determination of optimal intertemporal allocation of certain resources. In the available general theory, this insight has been articulated most forcefully by McKenzie [24, Introduction]; involving as it does a determinate, unit value of an important parameter in the rate of time preference, or alternatively, the degree of impatience, he has argued for the antecedent priority of the undiscounted case on the grounds of simplicity. We have yet to fully understand the robustness of this prescription[6], but here we establish a particularly strong version of it in the context of the two-sector RSS model.

Finally, as regards the importance of plane geometry, it is the case that the parameter $\xi_\sigma$ that somewhat incidentally emerged in the multi-sectoral analysis of the undiscounted case presented in [16], attains its determining role as the slope of the zero value-loss line (the von-Neumann facet) in the two-sector analysis presented in [14]. Such a line, together with the 45°-degree line and the cobweb diagrams associated with it, can be used to furnish a complete characterization of the optimal policies, both in the short- and in the long-run. Indeed, subsequent work in [15] and [45] has built on this analysis to identify $\xi_\sigma$ to be the crucial bifurcation parameter for the full RSS model; the geometrical identification of the optimal policy function can be used for a rigorous algebraic verification as is done in [15].[7] The simplicity of the two-sector version of the RSS model allows its reduced form expression to be diagrammed in the two-period (today-tomorrow) plane, and thereby the model's apparent intricacies rendered transparent by methods of plane geometry. To be sure, such diagrams are available in McKenzie's work[8], and used more explicitly by Boldrin-Deneckere, Nishimura-Yano and others to develop anti-turnpike theorems[9], but they had not been used previously, in of themselves, as an exclusive

[5] In addition to Ramsey [35] and Koopmans [19], see [9] for discussion and additional references.

[6] To be sure, McKenzie's prescription has not always been followed; see for example the textbook [42] and the monograph [20]. Majumdar-Nermuth [21] is a notable exception in presenting a unified analysis of the two cases.

[7] It should be noted that this has been accomplished in almost full measure but not completely; there still remains a claim about the optimal policy correspondence for the case $\xi = 1$ in [14] that has not been verified in [15].

[8] See the relevant figures in [23, 24, 25, 26].

[9] Such a theorem refers to the determination of parametrizations of two-sector models that generate chaotic dynamics; see [2] and [31, 32, 33]. For detailed discussion and additional references to the work of these authors, see [14, Section 10].

vehicle to obtain a complete characterization of the optimal policy correspondence in an undiscounted setting. More specifically, their promise remains yet to be investigated for the discounted setting.

In this essay, we focus on the second and third issue as a prelude for a fuller future investigation of the first, and since all of our methodological points can be fully articulated in a two-sector setting with a single type of machine, we limit ourselves to it.

We show that the geometrical apparatus developed in [14] for the undiscounted case with linear felicities carries over in almost "full measure" to the discounted case. In particular, part of the role of the 45°-degree line is taken over by a line with slope $1/\rho$, the zero-value loss line, now reckoned in terms of the modified golden-rule prices, and is shown to be independent of $\rho$. Furthermore, by retaining all of the features identified in the earlier analysis, it leads to the somewhat surprising conclusion that the von-Neumann facets and the modified golden-rule stock remain the *same* as in the undiscounted case. Thus, once these basic features of the geometry are in place, one can see at a glance why the undiscounted and discounted dynamics remain *identical* in all cases where $\xi$ lies in the range $-1 < \xi < 1$.[10] The case $\xi > 1$ proves more recalcitrant, and it is clear that its complete solution must have a recourse to analytical methods. Nevertheless, the outlines of a possible solution are offered by a consideration of specific cases amenable to a geometric representation and the computation of specific numerical examples; we offer a detailed analysis of one case within this regimen. The case $\xi = 1$ is easier, but it is also one in which the discounted and the undiscounted cases are *not* identical; therefore we also consider its analysis on its own. All in all, the geometry enables the undiscounted and discounted formulations to be put on the same table, so to speak, and leads to a satisfying unification of the analysis.

The rest of the essay proceeds as follows. Section 2 outlines the two-sector RSS model, and Section 3 recalls the basic results of the undiscounted case and the geometry that is used to prove them. Section 4 delineates the modifications that are required once discounting is introduced; and in particular, they lead to the the discovery of a line dual to the zero-value loss line $MV$ presented and discussed in [14]. This line determines the capital stock that gives rise to a two-period cycle[11], and is an important geometrical benchmark for the model. Section 5 uses the geometrical apparatus to give a complete characterization of the policy function for all values of $\xi$ in the interval $]-1, 1[$. Section 6 focusses on the parametrization $\xi(1-d) = 1$, a geometric situation in which two important lines are perpendicular, and section 7 the case $\xi = 1$,

[10] In the two-sector case with a single type of machine, there is no need for the subscript $\sigma$.

[11] While important for the subsequent analysis, the general importance of two-period cycle is discussed in [30].

another geometric situation in which two important lines are perpendicular. It is this perpendicularity that makes these two cases particularly amenable to geometrical analysis. Section 7 is the heart of the paper, at least from the point of view of economic substance. It shows the existence of a continuum of optimal 4-period cycles and the impossibility of chaotic behavior when the discount factor lies in a particular interval[12]. We conclude the essay with some summary observations regarding possible directions for further work.

## 2. The two-sector RSS model

A single consumption good is produced by infinitely divisible labor and machines with the further Leontief specification that a unit of labor and a unit of a machine produce a unit of the consumption good. In the investment-goods sector, only labor is required to produce machines, with $a > 0$ units of labor producing a single machine. Machines depreciate at the rate $0 < d < 1$. A constant amount of labor, normalized to unity, is available in each time period $t \in \mathbb{N}$, where $\mathbb{N}$ is the set of non-negative integers. Thus, in the canonical formulation surveyed in McKenzie (1986, 2002), the collection of production plans $(x, x')$, the amount $x'$ of machines in the next period (tomorrow) from the amount $x$ available in the current period (today), is given by the *transition possibility set*

$$\Omega = \{(x, x') \in \mathbb{R}^2_+ : x' - (1-d)x \geq 0 \text{ and } a(x' - (1-d)x) \leq 1\},$$

where $\mathbb{R}_+$ is the set of non-negative real numbers, $z \equiv (x' - (1-d)x)$ is the number of machines that are produced in the period $t$, and $z \geq 0$ and $az \leq 1$ respectively formalize constraints on reversiblity of investment and the use of labor. For any $(x, x') \in \Omega$, one can consider the amount $y$ of the machines available for the production of the consumption good, leading to a correspondence $\Lambda\colon\ \Omega \longrightarrow \mathbb{R}_+$ with

$$\Lambda(x, x') = \{y \in \mathbb{R}_+ : 0 \leq y \leq x \text{ and } y \leq 1 - a\,(x' - (1-d)x)\}.$$

The preferences of the planner are generally represented by a felicity function, $w\colon\ \mathbb{R}_+ \longrightarrow \mathbb{R}$, which is assumed to be continuous, strictly increasing and concave, and differentiable[13]. Finally, the *reduced form utility function*, $u\colon\ \Omega \longrightarrow \mathbb{R}_+$, is defined on $\Omega$ such that

[12] The eminent possibility of optimally chaotic trajectories for low enough discount factors is established in [17]. For further background, the reader can see [1], [5], [32].

[13] In this essay, we shall be working under the standing hypothesis that the felicity function $w(\cdot)$ is linear; these hypotheses are being presented as a general introduction to the two-sector RSS model.

$$u(x, x') = \max\{w(y) : y \in \Lambda(x, x')\}.$$

An *economy* $E$ consists of a triple $(\Omega, u, \rho)$, $0 < \rho \leq 1$ the discount factor, and the following concepts apply to it. A *program from* $x_o$ is a sequence $\{x(t), y(t)\}$ such that $x(0) = x_o$, and for all $t \in \mathbb{N}$, $(x(t), x(t+1)) \in \Omega$ and $y(t) \in \Lambda((x(t), x(t+1))$. A *program* $\{x(t), y(t)\}$ is simply a program from $x(0)$, and associated with it is a *gross investment sequence* $\{z(t+1)\}$ and a *consumption sequence* $\{c(t+1)\}$ as specified above. A program $\{x(t), y(t)\}$ is called *stationary* if for all $t \in \mathbb{N}$, $(x(t), y(t)) = (x(t+1), y(t+1))$. For all $0 < \rho < 1$, a program $\{x^*(t), y^*(t)\}$ from $x_o$ is said to be *optimal* if

$$\sum_{t=0}^{\infty} \rho^t \left[u(x(t), x(t+1)) - u(x^*(t), x^*(t+1))\right]$$
$$= \sum_{t=0}^{\infty} \rho^t \left[w(c(t+1)) - w(c^*(t+1))\right] \leq 0$$

for every program $\{x(t), y(t)\}$ from $x_o$. The case $\rho = 1$ will be referred to as the undiscounted case, and in this case, a program $\{x^*(t), y^*(t)\}$ from $x_o$ is called *optimal* if[14]

$$\liminf_{T\to\infty} \sum_{t=0}^{T} \left[u(x(t), x(t+1)) - u(x^*(t), x^*(t+1))\right]$$
$$= \liminf_{T\to\infty} \sum_{t=0}^{T} \left[w(y(t)) - w(y^*(t+1))\right] \leq 0$$

for every program $\{x(t), y(t)\}$ from $x_o$. A *stationary optimal program* is a program that is stationary and optimal.

The above expressions are routinely modified for the case of a linear felicity function. What is important and well-understood is that the linearity of $w(\cdot)$ does not imply the linearity of the reduced-form felicity function $u(\cdot, \cdot)$. The reduced-form model is now completely determined by the three parameters $(a, d, \rho)$.

## 3. Geometrical antecedents

In this section, we recall without proof the basic geometrical apparatus for the analysis of optimal programs in the undiscounted two-sector RSS model

[14] This is the overtaking criterion of Atsumi (1965) and von Weizsäcker (1965). Brock (1970) refers to our notion of optimality as *weakly maximal*; the reader is also referred McKenzie (1986) and Mckenzie (2002, p.256). Even though our primary emphasis is on the discounted case, the thrust of this essay is the essential complementarity of the two cases.

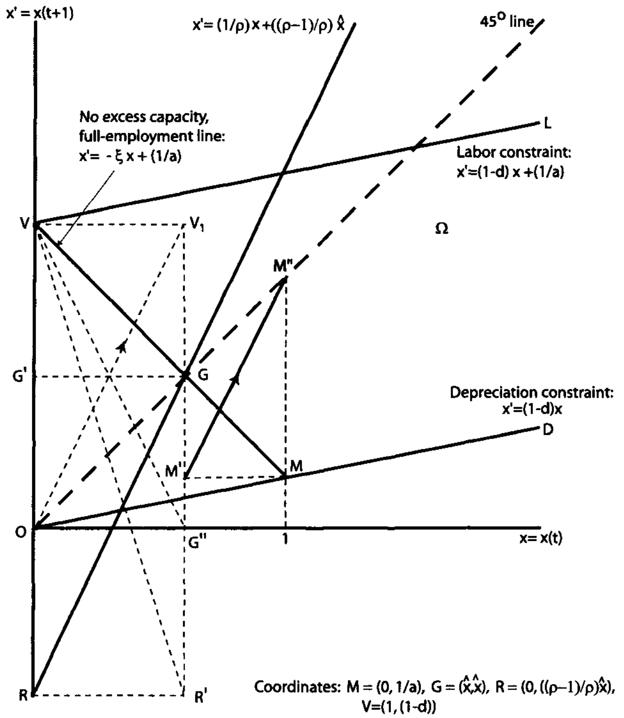


**Fig. 1.** Determination of the discounted golden-rule stock, the discounted golden-rule price system and the no value-loss line.

developed in [14]. Consider the today-tomorrow diagram furnished as Figure 1 and note that the transition possibility set $\Omega$ is given by the "open" rectangle $LVOD$. The indifference curves of the reduced form utility function $u(\cdot,\cdot)$ are kinked lines, the kink lying on $VM$, with a vertical arm below it and an arm with slope $d$ to the right of it, the levels of indifference increasing as we move southeast. Thus $OVL$ is the indifference curve with zero utility.

Given this specification of technology and preference, it is easy to see that the *golden-rule stock* $\hat{x}$ is given by $G$, the plan that yields the highest utility among all plans in the triangle bounded by $OV$, $VL$ and the 45°-degree line. More formally, it is the unique plan that satisfies

$$u(\hat{x},\hat{x}) \geq u(x,x') \text{ for all } (x,x') \in \Omega \text{ such that } x \leq x'.$$

Indeed, the geometry allows a quick algebraic determination of the value of the golden-rule stock in terms of the two parameters of the mode, $a$ and $d$. Focus on the line $VM$ in Figure 1, and note that

$$\xi = V_1 M' = \frac{VG'}{G'G} \Longrightarrow \frac{1}{a} - (1-d) = \frac{(1/a) - \hat{x}}{\hat{x}}$$
$$\Longrightarrow \hat{x} = \frac{1}{1+ad}$$
$$\Longrightarrow \frac{1}{a\hat{x}} = \frac{d(1+ad)}{ad} = \frac{d}{1-\hat{x}},$$

which implies that the slopes of the lines $M'M''$ and of $OV_1$ are identical.

The essential innovation in this treatment is the $MV$ line: it is not only a line delineating preferences — a preference-delineator line, so to speak — but it is also a locus of plans that represent full employment and full capacity-utilization, and as such, yield zero-value loss at the golden-rule price system $\hat{p}$. This is to say that $MV$ is the von-Neumann facet (of course, with the McKenzie facet, as named in [12], as a well-identified subset)[15]. Furthermore, the slope of $MV$ is given by $((1-\hat{p})/\hat{p})$, and lines parallel to it are constant value-loss lines, with higher values with movement away in either direction.

But these observations, along with elementary cobweb-type arguments, furnish a complete analysis of optimal growth in the model. When the initial capital stock is in the interval $[0, 1]$, and the $MV$ line has a slope in absolute value of less than or equal to one, any plan that begins on the $MV$ line stays on the $MV$ line, and thereby makes zero-value losses and is therefore optimal by Brock's theorem [3]. In these circumstances, it is easy to see that for all initial capital stocks in the range greater than unity, the minimum value-loss policy is to choose a plan on $MD$, and thereby again insure optimality. And so the only difficult issue concerns the case when the slope of the $MV$ line is greater than unity in absolute value. With the help of value-loss computations extending over at most three periods, it can be shown the optimal policy function in this case also has a horizontal segment — as the line $VGG_1D$ in Figure 4 below.

There is, however, another surprising result that goes beyond the computations of minimum value-loss trajectories. This is the identification of optimal trajectories for the case $\xi = 1$ that are *not* minimum value loss trajectories. This is the case of *indeterminacy* of optimal growth in the two-sector RSS model, a situation when an optimal policy correspondence, rather than a function, obtains. In terms of Figure 2, such a correspondence includes the triangle $GMG_1$ in addition to the two arms $VG$ and $G_1D$. The reason for what at first appears to be a surprising result becomes clear when the reader recalls that Brock's theorem, [3], only offers minimum value-loss as a sufficient condition for the optimality of a program. For the details of the geometric argument,

[15] Recall from McKenzie, [22], the von-Neumann facet to be the set of all plans in the transition possibility set, $\Omega$ in our case, that make zero value-loss at the golden rule prices, and in the discounted case, [23, 24], at the discounted golden-rule prices. The Mckenzie facet is a subset of the von-Neumann facet such that the program never leaves it *after* it enters it. For a formal treatment, [12].

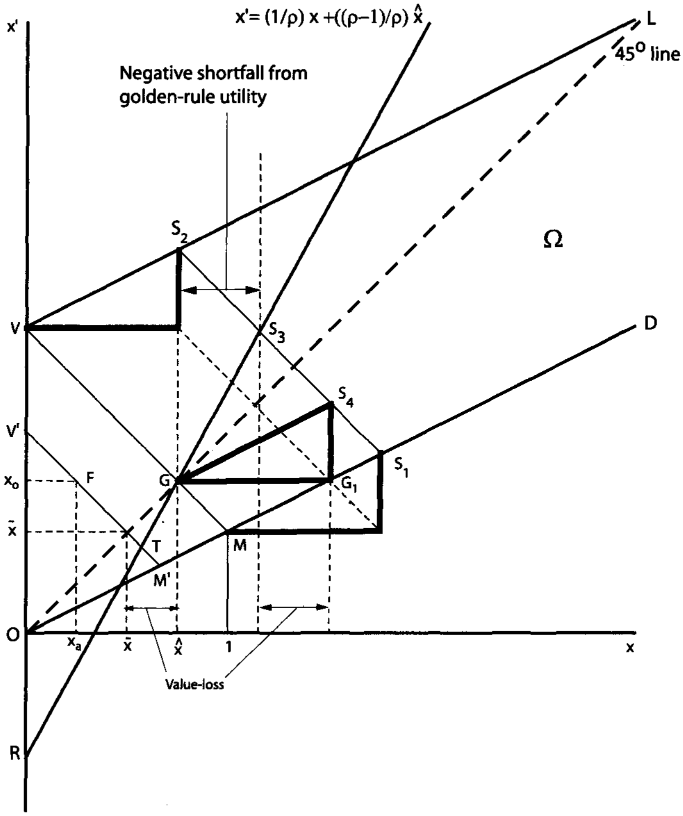


**Fig. 2.** Iso-value-loss lines.

the reader is referred to [14]. For a general theory of dynamic programming which establishes all those results analytically, and thereby complements the geometrical treatment, the reader can see [15].

In summary, it is the $MV$ line and its properties that make the geometric analysis viable in the undiscounted case, and the first methodological point to be made in this essay is that it is precisely these characteristics of the $MV$ line that extend to the discounted case in almost "full measure".

## 4. Geometry for the discounted setting

The geometry for the discounted case can perhaps be introduced best by the statement that part of the role of the 45°-degree line in the geometrical apparatus for the undiscounted case, its role relating to the computation of the value-losses at the golden-rule prices, is taken over by a line with slope $(1/\rho)$ and ordinate $((\rho-1)/\rho)\hat{x}$, $\hat{x}$ being the golden-rule stock. It is evident that with $\rho$ equal to one, this line reduces precisely to the 45°-degree line. We now turn to the details that implement this observation.

In Figure 1, keeping in mind the fact that the indifference curves of the reduced-form felicity function $u(\cdot,\cdot)$ and the transition possibility set $\Omega$ have nothing to do with the discount factor, impose on the undiscounted geometry a line $RG$ with slope $\rho^{-1}$ passing through $G$ and intersecting the $Y$-axis at the point $R$,[16] and the corresponding rectangle $ORR'G''$. Another entry into the geometry for the discounted case from the undiscounted one is through this rectangle. As the rate of time-preference and the degree of impatience become small, $\rho$ tends to one, and the rectangle tends to the the segment $OG''$.

To determine the discounted golden-rule stock, note that[17]

$$\frac{OR+GG''}{RR'}=\frac{OR+\hat{x}}{\hat{x}}=\rho^{-1}\Longrightarrow OR=((1-\rho)/\rho)\hat{x},$$

and hence the slope of the line $RG''$ (and the negative slope of the line $OR'$) is given by $(1-\rho)/\rho$. Once the slope and intercept of $RG$ are identified, we can obtain its equation as

$$x'=\rho^{-1}x+\rho^{-1}(\rho-1)\hat{x}.$$

But this allows the observation that the point $G$ yields the highest utility among all plans in $\Omega$ which lie "above" the line $RG$. More formally,

$$u(\hat{x},\hat{x})\geq u(x,x')\ \text{ for all }\ (x,x')\in\Omega\ \text{ such that }\ x\leq(1-\rho)\hat{x}+\rho x'.$$

Since $u(\hat{x},\hat{x})>u(0,0)$, $\hat{x}$ satisfies precisely the definition of the discounted golden-rule stock as in [6], [10, Definition 5] [23, 25] and [29]. This furnishes two results of consequence: first, the golden-rule stock is the discounted golden-rule stock, and second, as a consequence of the first, it is invariant to changes in the discount factor. These results are a direct consequence of the kinked indifference curves in our model, and also underscore the importance of viewing the discounted and undiscounted cases under one rubric. Note that this has allowed us to convert in the particular case of our simple model, a fixed-point problem into a maximization problem.

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[16] Whenever we refer to the line $RG$, we shall mean $RG$ extended at least till it intersects the line $ML$. Lines with slope of $\rho^{-1}$ play a prominent illustrative role in the diagrams of Lionel McKenzie ([25, Figure 2], [26, Figure 8]); here the line $RG$ is used as an engine of analysis. This line is referred to as $\rho_k^{-1}$ by McKenzie, $k$ being his notation for the discounted golden-rule stock (to be formally defined below); his line $\rho_0^{-1}$ is the line $OV_1$ in Figure 1, $RG$ shifted upwards to pass through the origin.

[17] We continue to abuse terminology by denoting a line and its length by identical notation.

The fact that the discounted golden-rule stock is a solution to a maximization problem in which the objective function $u$ maximized over the constraint set $\Omega$ and an additional (discounted) sustainability constraint, we follow [6] to appeal to Uzawa's version of the Kuhn-Tucker theorem [43] and obtain an associated shadow price $\hat{p}$ such that

$$u(\hat{x},\hat{x}) + (\rho-1)\hat{p}\hat{x} \geq u(x,x') + \hat{p}(\rho x' - x) \text{ for all } (x,x') \in \Omega. \quad (1)$$

We can now follow Radner [34] and define the value-loss $\delta^{\rho}_{(\hat{p},\hat{x})}(x,x')$ at the golden-rule price system $\hat{p}$ associated with the one-period plan $(x,x')$ by rewriting the above as[18]

$$\delta^{\rho}_{(\hat{p},\hat{x})}(x,x') = u(\hat{x},\hat{x}) + (\rho-1)\hat{p}\hat{x} - u(x,x') - \hat{p}(\rho x' - x) \text{ for all } (x,x') \in \Omega. \quad (2)$$

We turn to the determination of the golden-rule price system $\hat{p}$. Towards this end, consider Figure 1, and note that the zero net investment one-period plan $M$, given by $(1,(1-d))$, can be substituted in (1) to yield

$$\begin{aligned} &\hat{x} + (\rho-1)\hat{p}\hat{x} \geq 1 + \rho\hat{p}(1-d) - \hat{p} \\ &\Longrightarrow (\rho\hat{p})^{-1}(1-\hat{x}) \leq -(1-d) + \rho^{-1} + ((\rho-1)/\rho)\hat{x} \\ &\Longrightarrow (\rho\hat{p})^{-1} \leq \frac{d}{1-\hat{x}} - \frac{\rho-1}{\rho} = \frac{M''M}{M'M} + \frac{G''R'}{RR'}. \end{aligned} \quad (3)$$

By the same token, the maximal net investment, zero consumption one-period plan $V$, given by $(0, 1/a)$, can be substituted in (1) to yield

$$\hat{x} + (\rho-1)\hat{p}\hat{x} \geq \rho\hat{p}/a \Longrightarrow (\rho\hat{p})^{-1} \geq \frac{1/a}{\hat{x}} - \frac{\rho-1}{\rho} = \frac{OV}{OG''} + \frac{G''R'}{RR'}. \quad (4)$$

Since the lines $M'M''$ and $OV_1$ have identical slopes, we obtain

$$(\rho\hat{p})^{-1} = \frac{OV}{OG''} + \frac{G''R'}{RR'} = \frac{OV + G''R'}{OG''} = \frac{OV + OR}{RR'} = \frac{VR}{RR'}. \quad (5)$$

In terms of the geometry exhibited in Figure 1, the common slope of the lines $M'M''$ and $OV_1$ is negative of that of the line $VG''$. Now by (5), we know that $\rho\hat{p}$ is given by the difference in the slopes of $M'M''$ and $RG''$ which is equivalent to the (negative) of the difference in the slopes of $G''V$ and $OR'$. It is interesting that this difference is given by the slope of the line $VR'$. We now have the geometric characterization of the golden-rule price system that we seek.

$$\hat{p} = \frac{\rho\hat{p}}{\rho^{-1}} = \frac{RR'}{VR}\frac{G'R}{RR'} = \frac{G'R}{VR} \quad \text{and} \quad (1-\hat{p}) = 1 - \frac{G'R}{VR} = \frac{VG'}{VR}. \quad (6)$$

[18] We shall abbreviate $\delta^{\rho}_{(\hat{p},\hat{x})}(x,x')$ by $\delta^{\rho}(x,x')$.

It is thus clear that the ratios exhibited in (6) above depend on the discount factor $\rho$, and hence that the golden-rule price responds to $\rho$ even though the discounted golden-rule stock does not. What is interesting that the ratio of $(1-\hat{p})$ to $\rho\hat{p}$ is independent of $\rho$, and is identical to the slope of the $MV$ line. Formally,

$$\xi = \frac{VG'}{G'G} = \frac{VG'}{VR}\frac{VR}{RR'} = \frac{(1-\hat{p})}{\rho\hat{p}}. \tag{7}$$

This allows us to rewrite the equation of the line $MV$ as

$$\begin{aligned} x' = -\frac{1-\hat{p}}{\rho\hat{p}}x + C &\Longrightarrow \hat{x} = -\frac{1-\hat{p}}{\rho\hat{p}}\hat{x} + C \\ &\Longrightarrow \rho\hat{p}x' + (1-\hat{p})x = \hat{x} = u(\hat{x},\hat{x}). \end{aligned} \tag{8}$$

All that remains is the determination of the zero value-loss line, which is to say, the locus of all one-period production plans for which $\delta^\rho \equiv \delta^\rho_{(\hat{p},\hat{x})}(x,x') = 0$ where, from (2),

$$\begin{aligned} \delta^\rho &= u(\hat{x},\hat{x}) + (\rho-1)\hat{p}\hat{x} - u(x,x') - \hat{p}(\rho x' - x) \\ &= \big[u(\hat{x},\hat{x}) + (\rho-1)\hat{p}\hat{x} - x - \hat{p}(\rho x' - x)\big] + [x - u(x,x')] \\ &= \text{shortfall from golden-rule utility level} + \text{idle capacity}. \end{aligned} \tag{9}$$

Now for all points on the line $MV$, there is no idle capacity and therefore, on using the equation (8) above, we obtain the fact that there is zero value-loss. Proceeding in the converse direction, a zero value-loss implies that both terms in (9) are zero[19], and that therefore $(x,x')$ satisfy (8), and therefore constitute the line $MV$.

Finally, as in [14], we are now in a position to determine the value-loss of any one-period production plan, which is to say, of any point $(x,x') \in \Omega$. Lines parallel to $MV$ are indeed iso-value-loss lines, but they depict the value-loss after taking excess capacity into account. In Figure 2, consider a one-period production plan, say $F$ with coordinates $(x_a, x_o)$, in the surplus labor triangle $MOV$. There is no excess capacity of capital and hence its utility is furnished by its first coordinate, leading to the second term in (9) being zero. Hence its value-loss consists only of its shortfall from golden-rule utility level, the first term in (9). This is given by the difference between $\hat{x}$ and the abscissa of the point of intersection of the line $RG$ and a line $M'V'$ parallel to $MV$ and passing through $F$. Let the coordinates of this point of intersection $T$ be given by $(\bar{x}, \rho^{-1}\bar{x} + \rho^{-1}(\rho-1)\hat{x})$, and hence the equation of the line $M'V'$ is given by

$$x' = -\frac{1-\hat{p}}{\rho\hat{p}}x + C \Longrightarrow x' = -\frac{1-\hat{p}}{\rho\hat{p}}x + \frac{\bar{x}}{\rho\hat{p}} + \Big(\frac{\rho-1}{\rho}\Big)\hat{x}. \tag{10}$$

[19] Idle capacity is non-negative.

We can now obtain the shortfall from the golden-rule utility level that we seek. Since

$$\frac{\bar{x}}{\rho\hat{p}} = \left(x_o + \frac{1-\hat{p}}{\rho\hat{p}}x_a\right) - \left(\frac{\rho-1}{\rho}\right)\hat{x} \Longrightarrow \bar{x} = \rho\hat{p}x_o + (1-\hat{p})x_a - (\rho-1)\hat{p}\hat{x},$$

$$\begin{aligned} u(\hat{x},\hat{x}) - \bar{x} &= u(\hat{x},\hat{x}) + (\rho-1)\hat{p}\hat{x} - \rho\hat{p}x_o - (1-\hat{p})x_a \\ &= u(\hat{x},\hat{x}) + (\rho-1)\hat{p}\hat{x} - [x_a + \hat{p}(\rho x_o - x_a)] \\ &= \delta^{\rho}(x_a, x_o). \end{aligned} \tag{11}$$

In this demonstration, we have also shown that any one-period plan on $M'V'$ has the same value loss.

Next, we turn to one-period plans in the "open" parallelogram $LVMD$. In this case, value-loss stems from both excess capacity and from the negative shortfall from the discounted golden-rule utility level. We have already seen that this shortfall is the same for all plans on the line $S_1S_2$ parallel to $MV$, and is given by the difference between $\hat{x}$ and the abscissa of the point of intersection $S_3$ of $S_1S_2$ and the line $RG$. In order to show that $S_1S_2$ is an iso-value-loss line, all that remains is for us to show that the excess capacity associated with any one-period production plan on it, say $S_2$, $S_3$, $S_4$ or $S_1$, is identical. But this is easy from our procedure for computing excess capacity: all of the triangles with vertices $S_1$, $S_2$ and $S_4$ exhibited in Figure 2 are congruent, and hence their bases are equal.

Next, we show that the value losses increase as iso-value loss lines move "away" from the zero-value loss line $MV$ in either direction. This is clear when we limit ourselves to the full capacity, surplus labor triangle $MOV$. The difficulty concerning one-period plans in the full employment, excess capacity area $LVMD$ lies in the fact that as $MV$ moves outwards, both the negative shortfall from golden-rule utility as well as the excess capacity increase. However, the latter increases more than the former. To see this, consider the parallel lines $M'V'$ and $M''V''$ in Figure 3. The increase in the shortfall amounts to $x_1x_2$, whereas the increase in the excess capacity is the amount $W_1W_2$. To see that $W_1W_2$ is always greater than $x_1x_2$, draw a line $V'F$ parallel to the line $RG$, and simply observe that the difference in the abscissae of the points $F$ and $V'$ (which is $x_1x_2$ since triangles with vertices $F$ and $F'$ are congruent) is greater than $W_1W_2$. And this is always so by virtue of the fact that the slope of the line $RG$ is steeper than the slope of $OD$, (and of $VL$) which is another way of saying that the rate of depreciation $d$ and the discount factor $\rho$ are always less than unity.

We have now substantiated our claim that the geometric arguments presented here simply generalize those developed in [14] for the undiscounted case. We now simply work around the $RG$ line instead of the 45°-degree line. All that remains is to show that a path with a minimal value loss is an optimal path among all paths starting from the same initial capital stock. We present

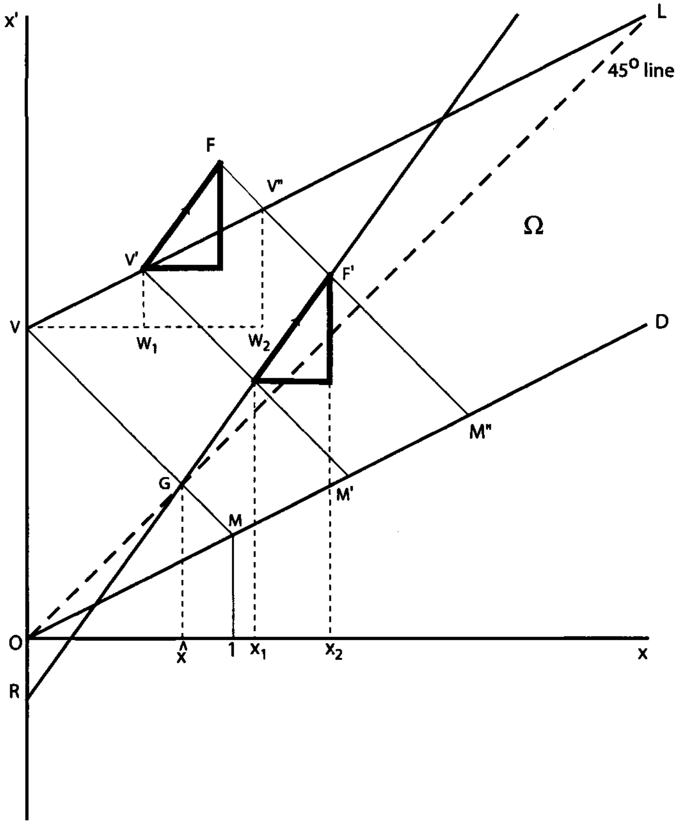


**Fig. 3.** Changes in value-loss as VM moves outwards.

a verification of this straightforward claim in the Appendix, and turn to the delineation of optimal programs.

## 5. The case $-1 < \xi < 1$

We are now in a position to apply the basic geometrical apparatus developed above to the case where the $MV$ line has a slope less than one in absolute value. For any initial stock in the unit interval, a standard cobweb exercise shows that a program chosen on the $MV$ line converges to the discounted golden-rule stock $G$; the reader can draw for herself a diagram analogous to Figure 1, or refer to Figures 8 and 9 and the associated text in [14]. But such a program makes a zero-value loss, and therefore by a transposition of Brock's argument to the discounted case, as presented in the Appendix, it is optimal. For initial capital stocks greater than one, there is value-loss but it is minimal for programs that choose on the relevant point of the $OD$ line, just as in [14]. We have thus shown the optimal policy functions in the case $-1 < \xi < 1$ to be independent of the discount factor $\rho$ and there to a unique optimal path with damped fluctuations in the case $0 < \xi < 1$ and monotonicity in the case $-1 < \xi \leq 0$.

This basic observation that the *feasibility* of full-employment and zero excess-capacity paths *throughout* time implies their optimality of course also extends to the case $\xi = 1$ with initial capital stocks in the $[1 - d, 1]$. However, since there is optimal policy correspondence in this case rather than an optimal policy function, and for other reasons to be made explicit in the sequel, we relegate it to a separate section.

We conclude this subsection with the observation, reminder really, that the independence of the optimal policy functions from the discount factor, also imply the independence of the von-Neumann and McKenzie facets from the discount factor. Furthermore, for all values of $\xi$ in the interval $]-1, 0]$, and for a computable threshold in the interval $]0, 1[$, the two types of facets are identical.

## 6. The case $\xi(1 - d) = 1$

In the case $\xi > 1$, there are no feasible paths with full-employment and zero excess capacity throughout, except the golden-rule point. So, we must have some value-loss along optimal paths starting from initial stocks other than the golden-rule stock. Thus, this is the interesting case, and it is now possible that discounting makes a difference. Future value losses are discounted compared to current ones, and so it might not be optimal to suffer the entire value loss in the initial period, unlike the undiscounted case. We focus on a specific parametrization.

### 6.1 The benchmarks

In order to get a geometric perspective on this case, consider Figure 4 in which $V'$ is a point such that the segment $OV' = OV = 1/a$. This implies that $\angle OVV' = \angle OV'V = 45°$-degrees, and that $\triangle OVC = \triangle OV'C$ are congruent right-angled triangles, where $C$ is the point of intersection of $VV'$ with the $45°$-degree line. Let the vertical from $C$ intersect the $X$-axis at $C'$ and the line $OD$ at $C_1$. Join $V'$ to $C_1$ and let its extension intersect the $Y$-axis at $C''$. Given the $45°$-degree angles, $\angle COV' = \angle CV'O$, $OC' = CC' = C'V'$ which identifies the capital stock $\tilde{x} = 1/2a$, and $\triangle OCC' = \triangle V'CC'$ as congruent, and

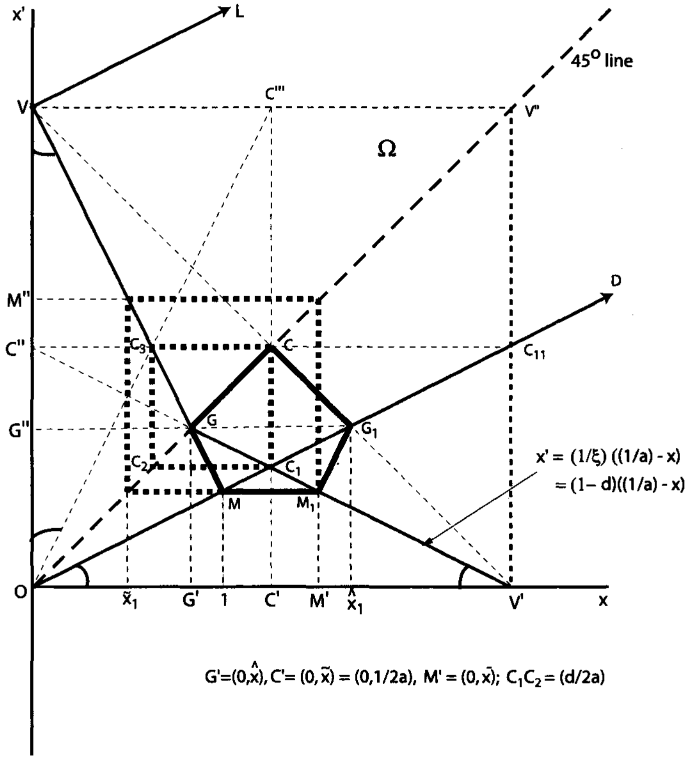


**Fig. 4.** Benchmarks in the case $\xi(1-d) = 1$ or $a = \xi(1+\xi^2)$ or $(1/a) = (1-d) + (1-d)^{-1}$

hence the slopes of the lines $OD$ and $V'C''$ as identical. Since $\triangle OC'C_1 \sim \triangle OV'C''$ and the segment $C'V'$ is half of $OV'$, the segment $C_1C'$ is also half of $OC'$, which is to say, that the length of $OC'$ is twice $(1-d)\tilde{x}$ which equals $((1-d)/a)$. This furnishes the equation of $V'C_1C''$, an important benchmark line dual to the line $OD$, as

$$x' = -(1-d)x + (1-d)/a = (1-d)((1/a) - x) \Longrightarrow x = \xi x' - 1/a. \quad (12)$$

It is important to note that so far we have not used the distinguishing characteristic of the case that we are considering; namely that $MV$ is perpendicular to $OD$. This translates geometrically into the fact that $\angle DOV' = \angle OVM$. [$\angle OVM$ is complementary[20] to $\angle VOD$ which is complementary to $\angle DOV'$.]

[20] We recall for the technically advanced reader the high-school terminology whereby complementary angles are two angles that sum up to a right angle.

Now let $G$ denote the intersection of the line $V'C''$ with the $45°$-degree line. We have to show that $G$ is the point that designates the discounted golden-rule stock, which is to say that it is also the intersection of $MV$ and the $45°$-degree line. Towards this end, we consider $\Delta VG''G$ and $\Delta V'GG'$. Since $\angle CVO = \angle CV'O$ and $\angle OVM = \angle GV'O$, $\angle CVG = \angle CV'G$ which implies that $\Delta CVG = \Delta CV'G$ which implies that $\Delta VG''G = \Delta V'GG'$ which implies that $GG'' = GG'$, which is to say that $G$ indeed designates the discounted golden-rule stock.

Now the full (and somewhat surprising) symmetry of the case under consideration becomes evident. Let the horizontal from $C$ intersect $MV$ at $C_3$, and the horizontal from $C_1$ intersect the $45°$-degree line at $C_2$. It is easy to see that $\Delta OCC_1$ is congruent to $\Delta OCC_3$, and that $\Delta OC_1C_2$ is congruent to $\Delta OC_3C_2$. Since $C$ and $C_2$ lie on the $45°$-degree line, the quadrilateral $CC_1C_2C_3$ is a square. This establishes $\tilde{x}$ as a benchmark initial stock which leads to 2-period cyclical path.

It is now easy to "complete the square" of side $OV$ with mid-points $C_{11}$, $C'$, $C''$ and $C'''$. This establishes the collinearity of $C''$, $C_3$, $C$ and $C_{11}$ on the one hand, and that of $C'''$, $C$, $C_1$ and $C'$ on the other. $C$ is a central point in that it is the center of the square $OV'V''V$. Note also the central pentahedron $CGMM_1G_1$, where $M_1$ be the point of intersection of $V'D'$ and the horizontal from $M$. Since $C\tilde{x}$ is a perpendicular bisector of $MM_1$, the capital stock $\bar{x}$ represented by the abscissa of $M_1$, is given by $(1/a) - 1$. $(\bar{x}, 0)$ is dual to the point $(1, 0)$ in the sense that it occupies the same position with respect to $V'$ that $(1, 0)$ occupies with respect to $V$. Another way of saying this is that it is dual in the same sense that $\hat{x}$ is dual to $\hat{x}_1$, and that $\tilde{x}$ is dual to itself. We leave it to the reader to show that the line $MV$ extends to $C'$, and that the line $M_1G_1$ extends to $V''$ on the one hand, and to $C'$ on the other. Finally, since $\Delta OMG = \Delta V'M_1G_1$, $\angle VM_1G_1$ is a right angle and the line $V'M_1$ is "dual" to the line $MV$.

We now use the benchmarks established in Figure 1 to identify a facet that is closely related to the McKenzie facet on the von-Neumann facet $MV$ in a way specified in Section 6.5 below. We redraw Figure 4 as Figure 5 in which the primary focus is on the lines $MV$ and $V'G$, and on the point $M_1$. Let the horizontal from $M_1$ (and from $M$) intersect the $45°$-line at $M_2$, and the vertical from it intersect the same line at $M_4$. Let the horizontal at $M_4$ intersect the line $MV$ at $M_3$, continuing on to $M''$. Join $M_3$ and $M_2$. We claim that $M_3M_2$ equals $M_1M_2$, and that consequently $M_1M_2M_3M_4$ is a square. For this demonstration, note that $\Delta CGV = \Delta CGV'$ implies that $\angle CGV = \angle CGV'$ which in turn implies that $\angle M_2GM_3 = \angle M_2GM_1$. Since $\angle OVG = \angle OV'G$ (a distinguishing characteristic of the case under consideration), and since $M_4$ lies on the $45°$-degree line, and therefore with its abscissa equal to its ordinate (equal to $\bar{x}$), $\Delta V'M_1\bar{x} = \Delta VM''M_3$. This implies that $VM_3 = V'M_1$

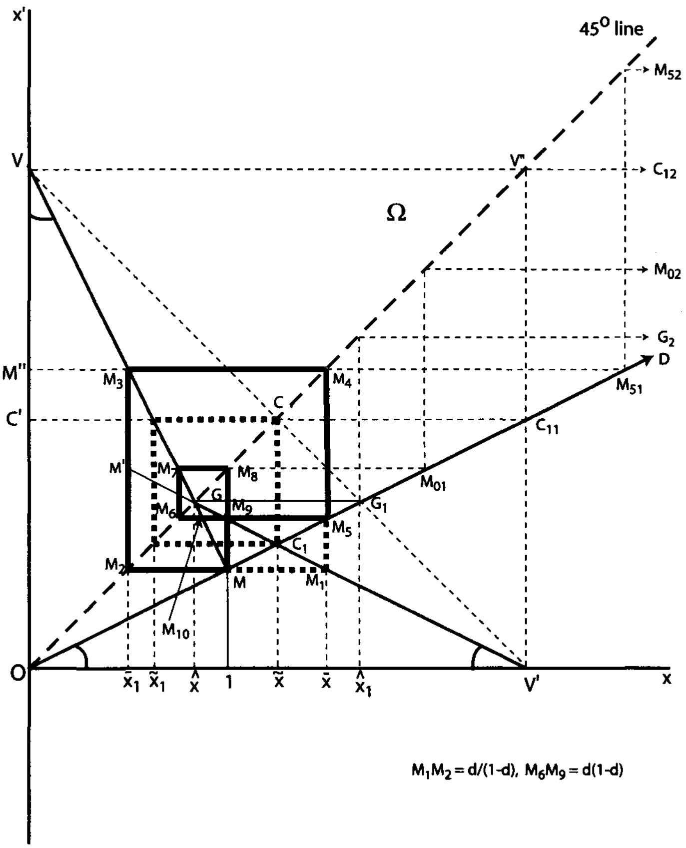


**Fig. 5.** Four-period cycles in the case $\xi(1-d) = 1$

which in turn implies that $GM_3 = GM_1$. Hence[21] $\triangle M_3M_2G = \triangle M_1M_2G$, and hence $M_3M_2$ equals $M_1M_2$. The demonstration is complete.

A byproduct of the above demonstration is the establishment of an important property of the line $V'G$. This is simply that any point on it can be used, in conjunction with the line $MG$ and the 45°-degree line to "complete a square." The length and the center of the square will vary with the vertex chosen, but the point $G$ will be an important invariant. Indeed, we have already seen two such

[21] Note that for the congruence of the triangles we are appealing to the criteria of the equality of two sides and that of the angle between them. Since the whole point of the exercise is to show that $M_3M_2$ is a vertical, we cannot appeal to the criterion of equality of one side and two angles. Our (successful) demonstration also proves that $M_3M_2$, and hence that $\angle M_2M_3G = \angle M_2M_3G$ which in turn equal $\angle OVG$ and $\angle OV'G$.

squares, with vertices $C_1$ and $M_1$. The argument can be easily abstracted and shown to rely on two congruences: $\triangle CGV = \triangle CGV'$, and the other between the triangles with vertices $V$ and $V'$. This establishes the consequence of two residual triangles and completes the demonstration of the figure being a square.

Now, let the horizontal from $M_5$ intersect the line $V'G$ at $M_9$. By the argument above, we can complete the square from $M_9$. Let its vertices be $M_6M_7M_8M_9$. We shall now show that this square is dual to the square $M_1M_2M_3M_4$, and that the square $C_1C_2C_3C$, identified in Figure 4, is dual to itself. But for this we need to first show that the vertical $M_8M_9$ is the same as the vertical through $M$. For this we have to show that $MM_1M_5M_9$ is a rectangle. Since[22] $\triangle MM_1M_5 = \triangle MM_1M_9, MM_1 = M_5M_9$ and $MM_5 = M_1M_9$. The demonstration of the rectangularity of the figure, and of the linearity of $MM_9M_8$ is complete.

Once we show the equality of the segments $M_7M_6$ and $M_1M_5$, we would have established a 4-period cycle for an initial capital stock of unity. But this equality is a simple consequence of the equality of $\triangle M_7M_6M_{10}$ and $\triangle M_{10}MM_9$. Note that in this cycle, a period of depreciation from 1 to $1-d$ ($M$ to $M_2$) is followed by a major investment program to $\bar{x}$ ($M_2$ to $M_1$) followed by another phase of depreciation, more substantial than the first, to $(1-d)\bar{x}$ ($M_5$ to $M_6$) followed by a final smaller investment phase from $(1-d)\bar{x}$ to unity ($M_6$ to $M_9$). It is now easy to check that the amplitude of the larger square is given by $d/(1-d)$, and that of the smaller by $d(1-d)$ resulting in a ratio of $1/(1-d)^2$. The average amplitude is given by $(1/2)d(1-d+(1/(1-d))=d/2a$, which is precisely the amplitude of the square with vertex $C_1$ identified in Figures 4 and 5.

All of this suggests an interesting interchange as we move the initial capital stock along $MM_1$ in Figure 5. For any point $m$ not equal to $C_1$ on the interval $MC_1$, there is a point $m'$ on the interval $C_1M_5$ in $V'M'$ such that a square with vertex $n$ intersects the interval $MM_5$ certainly at $m$, but also at another point $m'$. At the point $C_1$, the vertex of the relevant square, as well as its points of intersection with $MM_5$ are all $C_1$ itself. The projection of the interval $MM_5$ on the von-Neumann facet $MV$ is given by $MM_{10}$, and along with the point $G$, it has an interesting connection to the McKenzie facet that is adumbrated below.

So far there is no presumption that all these paths constitute the optimal paths or that the upper envelopes of the lines $MV$ and $OD$ constitute the optimal policy function. This obviously depends on the discount factor $\rho$ and we turn to this.

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[22] The criterion for the congruence of the two triangles is of one side and three angles.

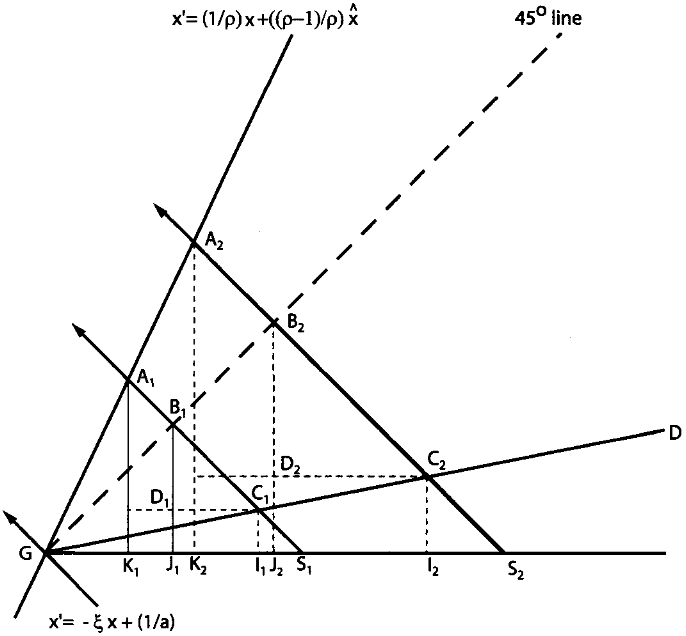


**Fig. 6.** Ratio of the value losses at $S_2$ and $S_1$

## 6.2 The uniqueness of $\hat{\rho}$

A singular feature of the particular model under consideration is that the golden-rule stock is independent of the discount factor, and the golden-rule price depends on it in the way delineated in Equation (7). We shall now exhibit another somewhat surprising property of the model; namely, that the ratio of the value-losses of two plans, $S_1 = (x_1, \hat{x})$ and $S_2 = (x_2, \hat{x})$ in Figure 6, where $x_1$ and $x_2$ are both greater than $\hat{x}$, is also independent of the discount factor.

Note from our previous discussion, and from a consideration of Figure 6, that the value-loss of $S_i$ at a zero discount factor is $I_iJ_i$, and at a discount factor of $\rho$ is $I_iK_i$; where $i = 1, 2$. By appealing several times to similar triangles, we obtain

$$\frac{I_1J_1}{I_2J_2} = \frac{D_1C_1}{D_2C_2} = \frac{B_1C_1}{B_2C_2} = \frac{GC_1}{GC_2} = \frac{GA_1}{GA_2} = \frac{A_1C_1}{A_2C_2} = \frac{I_1K_1}{I_2K_2}.$$

Indeed, as the discount factor goes to zero, $\rho^{-1}$ goes to infinity, the $RG$ line tends to the vertical and the value loss tends to excess capacity. This is brought out in the following

$$\frac{I_1J_1}{I_2J_2} = \frac{I_1K_1}{I_2K_2} = \frac{GC_1}{GC_2} = \frac{GS_1}{GS_2} = \frac{GI_1}{GI_2}. \tag{13}$$

We can now use these results to establish the existence and uniqueness of a discount factor at which a four period cycle and a path converging to the golden-rule stock in one period yield identical utility. Towards this end, consider the function

$$f(\rho) = \frac{\rho^2}{1-\rho^4} \quad \text{where } 0 \le \rho < 1. \tag{14}$$

It is easy to check that

$$f'(\rho) = \frac{2\rho(1+\rho^4)}{(1-\rho^4)^2} > 0 \text{ for all } 0 \le \rho < 1,$$

and that both

$$\lim_{\rho \to 1} f(\rho) = \lim_{\rho \to 1} f'(\rho) = \infty.$$

Now consider two paths each starting from an initial capital stock of unity in Figure 5. The first path involves a jump to the golden-rule stock where it stays, and the second goes from $M$ to $M_3$ to $M_5$ to $M_7$ back to $M$. Since $M$, $M_3$ and $M_7$ are all on the line $MV$, there is a loss in value only at the point $M_5$.

The value-loss for the first path is simply $\delta^\rho(1, \hat{x})$. The value loss for the second path is

$$\delta^\rho(\bar{x}, (1-d)\bar{x})\,(\rho^2 + \rho^6 + \cdots) = \rho^2/(1-\rho^4)\,\delta^\rho(\bar{x}, (1-d)\bar{x}).$$

Thus, the discount factor $\hat{\rho}$ that equates the value-losses of these two paths is given by the solution to the following equation.

$$\delta^\rho(1, \hat{x}) = \frac{\rho^2}{(1-\rho^4)}\,\delta^\rho(\bar{x}, (1-d)\bar{x})$$

$$\Longrightarrow f(\rho) = \frac{\rho^2}{1-\rho^4} = \frac{\delta^\rho(1,\hat{x})}{\delta^\rho(\bar{x}, (1-d)\bar{x})} \equiv \tau. \tag{15}$$

From the preceding discussion, we know that $f$ is a strictly increasing function over the unit interval with an asymptote at unity. We also know that the ratio of welfare losses is a constant function whose value is simply the ratio of excess capacity at the points $(1, \hat{x})$ and $(\bar{x}, (1-d)\bar{x})$. We thus obtain

$$\begin{aligned}\tau &= \frac{1 - [1 - a(\hat{x} - (1-d))]}{\bar{x} - 1} = \frac{a(\hat{x} - (1-d))}{\bar{x} - 1} = \frac{a\big(d - \frac{ad}{1+ad}\big)}{(1/a) - 2} \\ &= \frac{ad(1+ad-a)}{1+ad}\,\frac{a}{1-2a} = \frac{a^3 d}{(1-2a)(1+ad)}((1/a) - 1 + d) \\ &= -\frac{a^3 d}{(1-2a)(1+ad)}\xi = \frac{a^3 d}{(1-2a)(1+ad)(1-d)}\end{aligned} \tag{16}$$

The following equation for $\hat{\rho}$ is then a fundamental one.

$$\frac{\rho^2}{(1-\rho^4)} = \frac{a^3 d}{(1-2a)(1+ad)(1-d)} \Longrightarrow \rho^4 + \tau^{-1}\rho^2 - 1 = 0. \tag{17}$$

### 6.3 The optimal path

We begin with the case where the initial capital stock is $\tilde{x}$. Consider two alternative paths: the first where the planner moves to the golden-rule stock and stays there (the straight-down-the-turnpike path); and the second, the path in which the capital stock keeps returning to the initial capital stock after two periods. In terms of Figure 4, the path moves from $C'$ to $G$ compared to the periodic path from $C_1$ to $C_2$ to $C_3$ to $C_1$. We now determine the value of the discount factor $\hat{\rho}$ that equates the aggregate value losses of these two paths. This is to say that we want the root to the equation

$$\begin{aligned} \delta^{\rho}(\tilde{x},\hat{x}) &= \delta^{\rho}(\tilde{x},(1-d)\tilde{x})\,(1+\rho^2+\rho^4+\cdots) \\ \Longrightarrow \frac{\delta^{\rho}(\tilde{x},\hat{x})}{\delta^{\rho}(\tilde{x},(1-d)\tilde{x})} &= \frac{1}{1-\rho^2} \end{aligned} \tag{18}$$

Now, by Equation (13), and with reference to Figure 7, we obtain

$$\frac{\delta^{\rho}(\tilde{x},\hat{x})}{\delta^{\rho}(\tilde{x},(1-d)\tilde{x})} = \frac{GC_{11}}{GC_{12}} = \frac{GC_{11}}{GC_{11}-C_{11}C_{12}} = \left(1-\frac{C_{11}C_{12}}{GC_{11}}\right)^{-1}.$$

Now observe that

$$\frac{C_1C_{11}}{GC_{11}} = \frac{1}{\xi} \text{ and } \frac{C_1C_{11}}{C_{11}C_{12}} = \xi \Longrightarrow \frac{C_{11}C_{12}}{GC_{11}} = \frac{1}{\xi^2}.$$

We have thus shown that the root of Equation (18) is $(1/\xi)$.

Now consider another path alternative to the straight-down-the-turnpike path; namely, the path that moves from $C_{11}$ to $G$ in the $(t+2)^{th}$ period, $t \in \mathbb{N}$. It is of interest that the discount factor $\hat{\rho}$ that equates the aggregate value losses of these two paths is also $\xi$. To see this, we need to consider the root to the equation

$$\begin{aligned} \delta^{\rho}(\tilde{x},\hat{x}) &= \delta^{\rho}(\tilde{x},(1-d)\tilde{x})\,(1+\rho^2+\cdots+\rho^{2t}) + \rho^{2(t+1)}\,\delta^{\rho}(\tilde{x},\hat{x}) \\ \Longrightarrow \frac{\delta^{\rho}(\tilde{x},\hat{x})}{\delta^{\rho}(\tilde{x},(1-d)\tilde{x})} &= \frac{1}{1-\rho^2}. \end{aligned} \tag{19}$$

This simple result is of interest because it highlights circumstances in which a planner pursuing optimality has leverage as to when she can stop cycling. It leads to non-uniqueness in a way that has not been emphasized so far in this exposition.

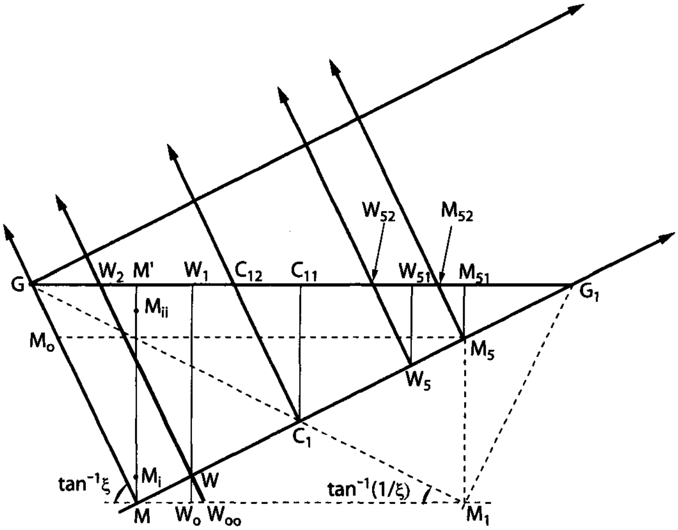


**Fig. 7.** Determination of $\hat{\rho}$ in the case $\xi(1-d)=1$

We now turn to the case where the initial capital stock is unity. Again, consider two alternative paths: the first where the planner moves to the golden-rule stock and stays there; and the second, the path that returns to the initial capital stock after four periods. In terms of Figure 5, the path that moves from $M$ to $G$ compared to the path $MM_3M_5M_7$. We now determine the value of the discount factor $\rho$ that equates the aggregate value losses of these two paths. This is to say that we want the root to the equation

$$\begin{aligned}\delta^\rho(1,\hat{x}) &= \delta^\rho(\bar{x},(1-d)\bar{x})\,(\rho^2+\rho^6+\cdots)\\ \Longrightarrow \frac{\delta^\rho(1,\hat{x})}{\delta^\rho(\bar{x},(1-d)\bar{x})} &= \frac{\rho^2}{1-\rho^4} = \left(\frac{1}{\rho^2}-\rho^2\right)^{-1}.\end{aligned} \tag{20}$$

Now, again by Equation (13), and with reference to Figure 7, we obtain

$$\frac{\delta^\rho(1,\hat{x})}{\delta^\rho(\bar{x},(1-d)\bar{x})} = \frac{GM'}{GM_{52}} = \frac{GM'}{GM_{51}-M_{51}V_{52}} = \left(\frac{GM_{51}}{GM'}-\frac{M_{51}M_{52}}{GM'}\right)^{-1}.$$

Next, consider $\triangle M_{52}M_5G_1$. By the distinguishing characteristic of the case that we are considering, this is right-angled triangle, and therefore has the property that[23]

$$\frac{M_{51}M_{52}}{G_1M_{51}} = \frac{1}{\xi^2}.$$

[23] This follows from the elementary argument that $M_{51}M_{52} = (1/\xi)M_5M_{51}$ and $G_1M_{51} = (\xi)M_5M_{51}$.

Again, by considering $\triangle GM_1G_1$, and appealing to the same property of right-angled triangles, we obtain

$$\frac{G_1M_{51}}{GM_{51}} = \xi^2.$$

Since $GM' = G_1M_{51}$, we have shown that the root of Equation (20) is $(1/\xi)$.

Next, consider the case where the initial capital stock is $\bar{x}$. Again, consider two alternative paths: the first where the planner moves to the golden-rule stock and stays there; and the second, the path that returns to the initial capital stock after four periods. In terms of Figure 5, the path that moves from $M_5$ to $G$ compared to the path $M_5M_7MM_3$. We now determine the value of the discount factor $\rho$ that equates the aggregate value losses of these two paths. This is to say that we want the root to the equation

$$\begin{aligned} \delta^\rho(\bar{x}, \hat{x}) &= \delta^\rho(\bar{x}, (1-d)\bar{x})\,(1 + \rho^4 + \rho^8 + \cdots) \\ \Longrightarrow \frac{\delta^\rho(\bar{x}, \hat{x})}{\delta^\rho(\bar{x}, (1-d)\bar{x})} &= \frac{1}{1-\rho^4}. \end{aligned} \tag{21}$$

Now, again by Equation (13), and with reference to Figure 7, we obtain

$$\begin{aligned} \frac{\delta^\rho(\bar{x}, \hat{x})}{\delta^\rho(\bar{x}, (1-d)\bar{x})} &= \frac{GM_{51}}{GM_{52}} = \frac{GM_{51}}{GM_{51} - M_{51}M_{52}} = \left(1 - \frac{M_{51}M_{52}}{GM_{51}}\right)^{-1} \\ &= \left(1 - \frac{M_{51}M_{52}}{G_1M_{51}}\frac{G_1M_{51}}{GM_{51}}\right)^{-1} = \left(1 - \frac{1}{\xi^4}\right)^{-1}. \end{aligned}$$

We have thus shown that the root of Equation (21) is $(1/\xi)$.

Next, consider the case where the initial capital stock is given by the abscissa of the point $W$ in Figure 7. By the argumentation presented above, there is a feasible path that returns to $W$ after 4 periods and is at $W_5$ after 2 periods, where $MW = W_5M_5$. We now determine the value of the discount factor $\rho$ that equates the aggregate value losses of this path as compared to the path that moves to the golden-rule stock in one period. This is to say that we want the root of the equation

$$\begin{aligned} GW_1 &= \frac{1}{1-\rho^4}\,GW_2 + \frac{\rho^2}{1-\rho^4}\,GW_{52} \\ &= \frac{1}{1-\rho^4}\,GW_2 + \frac{\rho^2}{1-\rho^4}(GM_{52} - W_{52}M_{52}) \\ &= \frac{1}{1-\rho^4}\,GW_2 + \frac{\rho^2}{1-\rho^4}(GM_{52} - GW_2) \\ &= \frac{1-\rho^2}{1-\rho^4}\,GW_2 + \frac{\rho^2}{1-\rho^4}\,GM_{52} \end{aligned} \tag{22}$$

We shall now show that the root of Equation (22) is $\xi$. Towards this end, we recall from Equation (20) that with $\rho = 1/\xi$,

$$\frac{\rho^2}{1-\rho^4} = \frac{GM'}{GM_{52}} \Longrightarrow \frac{\rho^2}{1-\rho^4} GM_{52} = GM'.$$

Furthermore, with $\rho = 1/\xi$, and again appealing to the property of right-angled triangles, we obtain

$$\frac{W_0W_{00}}{MW_0} = \rho^2 \Longrightarrow \frac{MW_{00}}{MW_0} = 1 + \rho^2 \Longrightarrow \frac{MW_0}{MW_{00}} = \frac{M'W_1}{GW_2} = \frac{1}{1+\rho^2}.$$

Putting these expressions together, we see that the right-hand side of Equation (22) is given by $GM'$ plus $M'W_1$ which is precisely equal to the left-hand side $GW_1$.

We now return to the case where the initial capital stock is unity but compare the straight-down-the-turnpike path with one that begins at the point $M_i$ in Figure 7 and returns to (say) $W$ after four periods. We now determine the value of the discount factor $\rho$ that equates the aggregate value losses of these two paths. This is to say that we want the root to the equation

$$GM' = \frac{1}{1-\rho^4} GW_2 + \frac{\rho^2}{1-\rho^4} GW_{52} - M_iW$$

which in turn implies that we seek the root of the equation

$$GW_1 = \frac{1}{1-\rho^4} GW_2 + \frac{\rho^2}{1-\rho^4} GW_{52}.$$

But this is precisely Equation (22), and we have already determined one of its roots to be $(1/\xi)$.

We now remain with the case where the initial capital stock is unity but compare the straight-down-the-turnpike path with one that begins at the point $M_{ii}$ in Figure 7. The distinguishing characteristic of this case is that in the initial periods the second path stays in the segment $M_6G$ of the von-Neumann facet and sustains no value-losses. Furthermore, depending on the proximity of $M_{ii}$ to $M'$ (the proximity of the ordinate of $M_{ii}$ to the golden-rule stock), the number of these initial periods, even though finite, can be arbitrarily large. In any case, since $\xi > 1$, there is a first time period at which the value of the capital stock of this path is greater than or equal to unity, and less than or equal to the abscissa of $M_5$. Without loss of generality, let this value be given by the abscissa of $W$. We now determine the value of the discount factor $\rho$ that equates the aggregate value losses of these two paths. This is to say that we want the root to the equation

$$GM' = \frac{1}{1-\rho^4} GW_2 + \frac{\rho^2}{1-\rho^4} GW_{52} - M_6 W$$

which in turn implies that we seek the root of the equation

$$GM' = \frac{1}{1-\rho^4} GW_2 + \frac{\rho^2}{1-\rho^4} GW_{52}.$$

But this is again precisely Equation (22), and we have already determined one of its roots to be $(1/\xi)$.

### 6.4 The optimal policy function

Define the following two real valued functions on $\mathbb{R}_+$:

$$h(x) = \max\left[-\xi x + (1/a), \hat{x}, (1-d)x\right]$$
$$g(x) = \max\left[-\xi x + (1/a), (1-d)x\right].$$

Let $\hat{\rho} = (1/\xi)$; then the optimal policy correspondence is given by

$$x(t+1) \in \begin{cases} \{x \in \mathbb{R}_+ : h(x(t)) \leq x \leq g(x(t))\} & \text{when } \rho = \hat{\rho} \\ h(x(t)) & \text{when } 1 \geq \rho > \hat{\rho} \\ g(x(t)) & \text{when } 0 < \rho < \hat{\rho} \end{cases}$$

### 6.5 Some additional facets

We return to Figure 5, and in particular to the policy function $VMD$, optimal for values of the discount factor less than $(1/\xi)$. In the discussion at the end of Section 6.1, we noted the existence of a point $n$ on $M_1C_1$ and two corresponding points, $m$ and $m'$ on $MG_1$ which are dual in the precise sense that a square with vertex $n$ intersects $OD$ at the points $m$ and $m'$. And as $n$ moves continuously between $M_1$ and $C_1$, the points $m$ and $m'$ move continuously from $M$ and $M_5$ on $OD$ towards $C_1$ and yield a continuum of 4-period cycles. These properties are a testament to the symmetry of Figure 4, something not as transparently apparent when one considers the (algebraic) equality $x(1-d) = 1$. Hence also the subset $MM_5$ of the graph of the optimal policy function, as well as its projection $MM_{10}$, (in short, the rectangle $MM_1M_5M_9$ in Figure 5), is of considerable interest.

Note, to begin with, that the interval $M_7M_3$ is not the McKenzie facet, even though it is a subset of the von-Neumann facet $VM$. The reason is straightforward; the optimal program steps out of it for one period after having stayed in it for two subsequent periods, and it does so for all periods of time. As such, we shall refer to it as the $m_1$*-facet*. The $m_0$*-facet* is indeed the McKenzie facet, and in principle, we can define an $m_i$*-facet*, for all $i \geq 0$.

However the segments $MC_1M_5$ and $M_{01}C_{11}M_{51}$ on the $OD$ line, more relevantly, on the graph of the optimal policy function, are also akin to the $m_1$-*facet*. They represent plans that reach the $m_1$-*facet* in one period, but since they do not lie on the von-Neumann facet cannot be designated as constituting an $m_2$-*facet*. We shall refer to them as constituting an $m_1(1)$-*facet*. But now the procedure is clear: just as the interval $M_{01}C_{11}M_{51}$ is a "stretching" of the $MC_1M_5$ interval, $M_{02}C_{12}M_{52}$ is a "stretching" of the $M_{01}C_{11}M_{51}$ interval and can be said to constitute the $m_1(2)$-*facet*. And similarly for the $m_1(i)$-*facet* as having been constituted by the interval $M_{0i}C_{1i}M_{5i}$ on $OD$, for all $i > 0$.

The alert reader has surely noticed the gaps $M_5G_1$, $M_{51}G_2, \ldots, M_{5i}G_{i+1}$ on the graph of the optimal policy function represented by the line $OD$. It makes sense to refer to the interval $M_{5i}G_{i+1}$ as the $v(i)$-facet, where $i \geq 1$; a plan on the $v(i)$-facet is not on the von-Neumann but can reach it in $i$ periods. Thus, the von-Neumann facet is really the $v(0)$-facet. There are however points on the von-Neumann facet that can reach a particular $v(i)$-facet in one period. We shall refer to them, analogous to our treatment above, as constituting the $v_i$-facets. In this case, the index $i$ has an upper bound.

It is important to note that these benchmarks, and the decompositions of the von-Neumann facet associated with them, can be used for a detailed examination of the von-Neumann facet in other examples, and most preferably, in the general situation.

## 7. The case $\xi = 1$

In [14], the claim is made that the optimal policy correspondence for the undiscounted RSS model for the case $\xi = 1$ is given by[24]

$$x(t+1) \in \begin{cases} \{x \in \mathbb{R}_+ : -\xi(x(t)) + (1/a) \leq x(t) \leq (1-d)(x(t))\} & \\ & \text{when } \hat{x} \leq x(t) \leq \hat{x}/(1-d) \\ -\xi(x(t)) + (1/a) & \text{when } 0 \leq x(t) < \hat{x} \\ (1-d)(x(t)) & \text{when } x(t) > \hat{x}/(1-d) \end{cases}$$

In the undiscounted case, one should have expected non-uniqueness of optimal paths for $\xi = 1$ on the ground that optimal behavior ought to be continuous in the parameter values. That is, the cyclic optimal path for $\xi = 1$ is just the limiting case of the unique optimal path with damped fluctuations for $\xi < 1$. And the straight down the turnpike optimal path for $\xi = 1$ is the limiting case of the unique straight down the turnpike optimal path for $\xi > 1$.

[24] Also see the partial verification of this claim in [15].

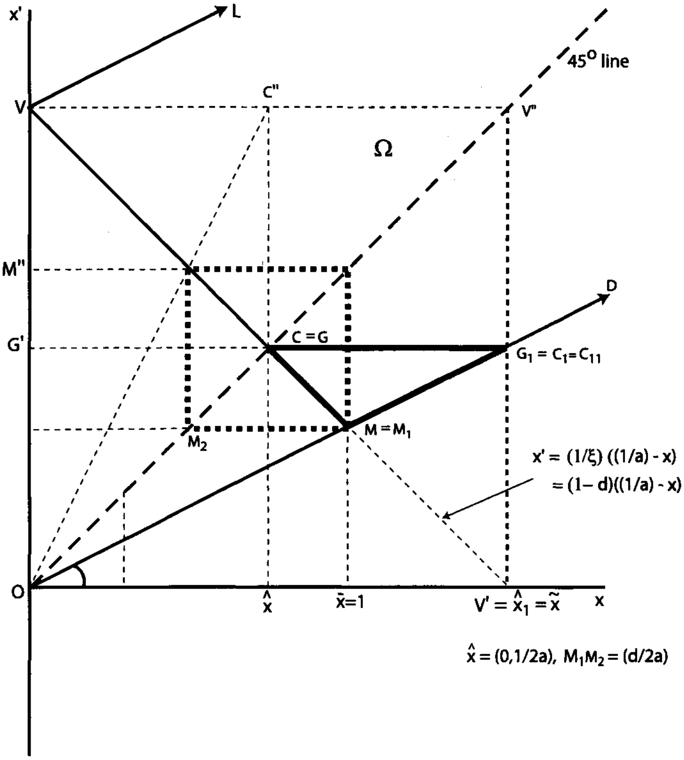


**Fig. 8.** Benchmarks in the case $\xi = 1$

We leave it as a routine exercise for the reader to apply the geometrical apparatus developed here to show that the optimal policy function is given by the "check map" $VMD$ in Figure 8, and that one obtains persistent symmetric fluctuations, 2-period cycles, from initial stocks in $[1 - d, 1]$). The non-uniqueness issue cannot arise for $\xi = 1$ in the discounted case, because the transversality condition takes care of the "terminal capital stocks terms" always, whereas they continued to be a factor to take into account even asymptotically in the undiscounted case. Thus, we must have uniqueness for all $\rho$ less than one, and then the continuum (which would of course include the persistent symmetric fluctuations path) when $\rho$ equals one. There is nothing in the model to disturb upper hemicontinuity of the correspondence. It is thus of interest that the only difference between the discounted and undiscounted cases lies in that there is no possibility of non-uniqueness of optimal paths for the case $\xi = 1$.

We also leave it to the reader to give a complete decomposition of the graph of the optimal policy correspondence in terms of the von-Neumann and

McKenzie facets, as well as the affiliated facet concepts developed in Section 6.5 above. The ease and completeness with which this can be done suggests perhaps the analytical use and viability of the facet concepts we formulate motivated by the case $\xi(1-d)=1$.

However, there is an aspect of the case $\xi = 1$ that is brought out by the geometrical treatment offered in this essay. This is the interplay between it and the case $\xi(1 - d) = 1$ that gives insight into both, and in particular allows us to see the second as a degenerate specialization of the first. Note that the distinguishing characteristic of the first case, as brought out in Figure 4, is that the line $V'M'$, dual to the line $VM$, has the same slope, in absolute terms, as the line $OD$. Or to put the matter in terms of the $OD$ line, its dual line $OC''$ has the same slope as the $MV$ line. To put the matter yet another way, the triangle $\triangle OC_3V$ and $\triangle OC_1V'$ are both isosceles triangles whose larger side is precisely equal to the side of the square $V'OVV''$.

It is these symmetries that are inherited by the geometry of the case $\xi = 1$ presented in Figure 8. Figure 8 is simply the case where the lines $MV$ and $M'V'$ are collinear. In terms of an algebraic presentation,

$$x' = -\xi x + (1/a) \Longleftrightarrow x' = (1/\xi)((1/a) - x) \Longrightarrow \xi = 1.$$

Thus the case $\xi = 1$ can be seen as the limit of a procedure whereby the point $G$ moves to the point $C$. Such a procedure is visually evident by looking at Figures 4 and 8 together: in Figure 8, $M$ moves to $M_1$, $C_1$ is pulled up to $G_1$, and $C_{11}$ is pulled down to $G_1$. Thus the kinked two-line segment $GCG_1$ in Figure 4 collapses to the segment $GG_1$ in Figure 8, the kinked two-line segment $GMM_1$ in Figure 4 collapses to the segment $GM$ in Figure 8, and finally, the segment $M_1G_1$ in Figure 4 remains the same $M_1G_1$ segment in Figure 8. In summary, the pentahedron $MM_1G_1CG$ of Figure 4 is transformed to the triangle $MCG_1$ of Figure 8. Now, the two period cycle emanating from $C_1$ in Figure 8 becomes the discounted golden-rule stock, and hence the 0-period cycle, and the and the continuum of four period cycles degenerates to a continuum of two period cycles!

## 8. Concluding observations

It bears emphasis that despite all this work, the characterization of an optimal program in the two-sector RSS model with discounting remains essentially open in the case when $\xi$ is greater than unity. We have only considered two values in this continuum. However, through an exploitation of their particular structural characteristics, we have provided a complete analysis for those two values. In terms of economic substance, we have shown the existence of a continuum of four-period cycles in case $\xi(1 - d) = 1$, and a continuum of two-period cycles in case $\xi = 1$. Indeed, we have substantiated the sense in which we view the second as a degenerate case of the first. For both cases, we have furnished a value for the threshold discount factor above which the optimal policy functions, and hence the optimal transition dynamics, remain identical between the discounted and the undiscounted cases. Furthermore, in addition to the delineation of the von-Neumann and McKenzie facets, we have conceptualized and computed subsets of the graph of the optimal policy function, the so called $m_i$-facets and $v_i$-facets, that can be seen as their natural extensions. One of the important consequences of this completed analysis is that, unlike the case in [17], it rules out the possibility of chaos in the two particular cases we consider, no matter how small the discount factor.

In terms of the contribution to the geometrical analysis, the fact that the discounted golden-rule stock is independent of the discount factor, that the $MV$ line remains the zero value-loss line even at the discounted golden-rule prices (which do depend on the discount factor) and retains all its properties, are pleasant facts which make the geometry viable. But perhaps the most important has been the discovery of the $M'V'$ line dual to the $MV$ line, obtained by "completing" the relevant square. The fact that its intersection with the $OD$ line yields the capital stock at which there are two-period cycles is an important benchmark. It also holds generally; that is to say, for all values of $\xi > 1$.

It seems clear, however, that the complete characterization of the optimal policy for the entire case $\xi > 1$ is a difficult problem despite, perhaps because of, a model consisting of only three numbers $(a, d, \rho)$, two of which lie in the unit interval. Because of the lack of differentiability, the possibility of a kink in the optimal policy function, assuming it is a function, precludes the application of the Euler-Lagrange variational equalities generally available in the calculus of variations; see [29, 30] and his references. Further progress will be had on a case by case basis involving geometry certainly, but possibly also numerical specifications to build up one's intuition. We intend to proceed along this path in future work.

## 9. Appendix

Let $\mathbb{N}$ be the set of positive integers, and $\mathbb{N}' = \mathbb{N} \cup \{0\}$. Consider any two feasible paths $\{x'(t)\}_{t\in\mathbb{N}'}$ and $\{x''(t)\}_{t\in\mathbb{N}'}$. Then for any $T \in \mathbb{N}$, we obtain by appealing to (2) above,

$$\sum_{t=0}^{T} \rho^t (u(x'(t), x'(t+1)) - u(x''(t), x''(t+1))$$

$$= \sum_{t=0}^{T} \left[\rho^{t+1}\hat{p}(x''(t+1) - x'(t+1)\right] + \left[\rho^t \hat{p}(x'(t) - x''(t))\right]$$

$$+ \left[\rho^t(\delta^\rho(x''(t), x''(t+1) - \delta^\rho(x'(t), x'(t+1))\right.$$
$$= \left[\rho^{T+1}\hat{p}(x''(T+1) - x'(T+1)\right] + \left[\hat{p}(x'(0) - x''(0)\right]$$
$$+ \sum_{t=0}^{T} \rho^t(\delta^\rho(x''(t), x''(t+1) - \delta^\rho(x'(t), x'(t+1)).$$

Since $x'(t)$ and $x''(t)$ lie in a bounded set for all $t \in \mathbb{N}'$, (see [16]), and since $0 < \rho < 1$, we obtain, for the case $x'(0) = x''(0)$,

$$\sum_{t=0}^{\infty} \rho^t(u(x'(t), x'(t+1)) - u(x''(t), x''(t+1))$$
$$= \sum_{t=0}^{\infty} \rho^t(\delta^\rho(x''(t), x''(t+1) - \delta^\rho(x'(t), x'(t+1)).$$

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